UNIT - V

Moment of Inertia:

Moment of inertia of a particle of mass 'm' about a straight line is defined to be mr^2, where r is perpendicular distance of the particle from the straight line.

The moment of inertia of a system of particle of masses $m_1, m_2, \ldots, m_n$ at distance $r_1, r_2, \ldots, r_n$ from the straight line is

$$m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2$$

Problem:

1. Find the moment of inertia of the circular ring.

Soln:

Let the ring be divided into particles of masses $m_1, m_2, m_3, \ldots m_n$

Then their moment of inertia about l are $m_1 a^2, m_2 a^2, \ldots m_n a^2$

$$\text{M.I of the circular ring} = m_1 a^2 + m_2 a^2 + \ldots + m_n a^2$$
$$= (m_1 + m_2 + \ldots + m_n) a^2$$
$$= Ma^2$$

2. Find the M.I of Right circular hollow Cylinder.

Soln:

Divide the Cylinder into thin circular rings. whose planes are perpendicular to l.

The masses of rings are $m_1, m_2, \ldots m_n$

M.I of circular rings are $m_1 a^2 + m_2 a^2, \ldots m_n a^2$

$$\left.\begin{array}{l}\text{M.I of the right Circular} \\ \text{hollow Cylinder about } l\end{array}\right\} = m_1 a^2 + m_2 a^2 + \ldots + m_n a^2$$

$$= (m_1 + m_2 + \ldots + m_n) a^2$$

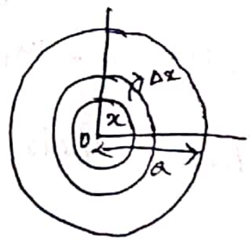
$$= M a^2$$

3. Find the M.I of Circular lamina

Soln:

Divide the circular lamina into thin Concentric Circular rings

Consider the ring whose inner and outer radii are $x, x+\Delta x$



For this ring,

$$\text{Area} = 2\pi x \,.\, \Delta x$$

$$\text{Mass} = \text{Area} \times \text{Mass per unit area}$$

$$= 2\pi \; x \Delta x \times \frac{M}{\pi a^2}$$

$$\text{Mass} = \frac{2Mx}{a^2} \Delta x$$

Scanned with CamScanner

$$\text{M.I about } l = \left(\frac{2Mx}{a^2}\Delta x\right)x^2$$

$$= \frac{2Mx^3}{a^2}\Delta x$$

$$\left.\begin{array}{l}\text{M.I of circular}\\ \text{lamina about } l\end{array}\right\} = \int_0^a \frac{2Mx^3}{a^2}\,dx$$

$$= \frac{2M}{a^2}\left(\frac{x^4}{4}\right)_0^a$$

$$= \frac{2M}{a^2}\left(\frac{a^4}{4}\right)$$

$$\left.\begin{array}{l}\text{M.I of circular lamina}\\ \text{about } l\end{array}\right\} = \frac{Ma^2}{2}$$

4. Find the M.I of Solid right Circular Cylinder.

Soln:

Divide the cylinder into thin circular lamina $\perp^r$ to the axis of the Cylinder

If their mass are $m_1, m_2, \dots m_n$

M.I of circular lamina's are $\frac{m_1a^2}{2}, \frac{m_2a^2}{2}, \dots \frac{m_na^2}{2}$

$$\left.\begin{array}{l}\text{M.I of the solid right circular}\\ \text{cylinder about } l\end{array}\right\} = \frac{m_1a^2}{2} + \frac{m_2a^2}{2} + \dots + \frac{m_na^2}{2}$$

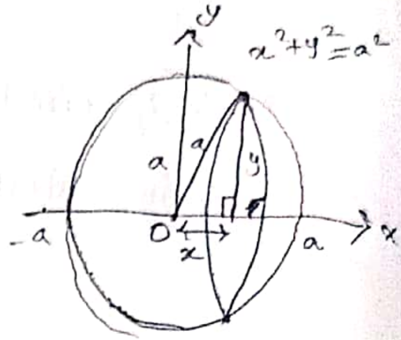
$$= \frac{a^2}{2}(m_1 + m_2 + \dots + m_n)$$

$$= \frac{Ma^2}{2}$$

5. Find the M.I of Solid Sphere

Soln:

Divide the sphere into thin circular lamina $\perp$r to the diameter.



Consider the lamina whose distance from the centre of the sphere is x

Let,

radius of the sphere $= a$

radius of the disc $= y$

$$x^2+y^2 = a^2$$

$$y^2 = a^2-x^2$$

$$y = \sqrt{a^2-x^2}$$

Area of the disc $= \pi y^2 dx$

Mass of the disc $= \pi y^2 \rho dx$

M.I of the disc $= (\pi y^2 \rho dx) y^2/2$

M.I of the solid sphere $= \int_{-a}^{a} \pi \rho \frac{y^4}{2} dx$

$$= \frac{2\pi\rho}{2} \int_0^a y^4 dx$$

$$= \frac{2\pi\rho}{2} \int_0^a (a^2-x^2)^2 dx$$

$$= \pi\rho \int_0^a (a^4+x^4-2a^2x^2) dx$$

$$= \pi\rho \left(a^4x + x^5/5 - 2a^2 x^3/3\right)_0^a$$

$$= \pi\rho \left(a^5 + \frac{a^5}{5} - \frac{2a^5}{3}\right)$$

$$= \pi\rho \left(\frac{15a^5 + 3a^5 - 10a^5}{15}\right)$$

$$= \pi\rho \left(\frac{8a^5}{15}\right)$$

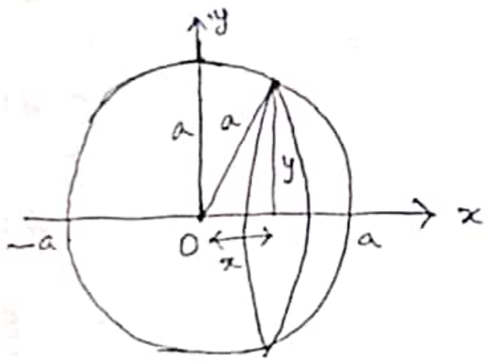
$$= \pi \left(\frac{8a^5}{15}\right)\left(\frac{M}{4/3\,\pi a^3}\right)$$

$$= \pi\, \frac{8a^5}{15} \times \frac{M}{4\pi a^3} \times 3$$

M.I of the solid sphere $= \frac{2}{5} Ma^2$

6. Find the M.I of hollow sphere.

Soln:



Radius of sphere $= a$

Radius of circular ring $= y$

Equation of circular ring.

$$x^2 + y^2 = a^2$$

$$y^2 = a^2 - x^2$$

Area of circular ring $= 2\pi y\, ds$

Mass of circular ring $= (2\pi y\, ds)\rho$

M.I of circular ring $= (2\pi y\, ds\, \rho)y^2$

$$= 2\pi\rho\, y^3\, ds$$

M.I of hollow sphere $= \int_{-a}^{a} 2\pi \rho y^3 ds$

$= 2\pi\rho \int_{-a}^{a} y^3 ds$

$= 2\pi\rho \times 2 \int_{0}^{a} y^3 \sqrt{1+y'^2}\, dx$

$= 4\pi\rho \int_{0}^{a} y^3 \sqrt{1+\left(-\frac{x}{y}\right)^2}\, dx$

$= 4\pi\rho \int_{0}^{a} y^3 \sqrt{\frac{y^2+x^2}{y^2}}\, dx$

$= 4\pi\rho \int_{0}^{a} y^3 . \frac{a}{y}\, dx$

$= 4\pi\rho a \int_{0}^{a} y^2\, dx$

$= 4\pi a\rho \int_{0}^{a} (a^2-x^2)\, dx$

$= 4\pi a\rho \left[a^2x - \frac{x^3}{3}\right]_0^a$

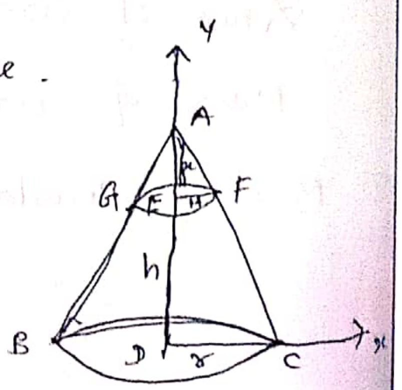
$= 4\pi\rho a \left[a^3 - \frac{a^3}{3}\right]$

$= 4\pi\rho a \left(\frac{2a^3}{3}\right)$

$= \frac{8\pi a^4}{3} \left(\frac{M}{4\pi a^2}\right)$

$$\left.\begin{matrix}\text{M.I of hollow} \\ \text{sphere}\end{matrix}\right\} = \frac{2Ma^2}{3}$$

7. Find the M.I of Solid Cone.



Soln:

A - vertex of cone

h - height of cone

x, y – Two axis along AD, perpendicular to AD respectively.

Divide the Cone into a number of Circular slices of thickness dx.

Let G, F be one such slice of radius y and at a distance x from A.

Areal of the disk $= \pi y^2 dx$

Mass of the disc $= \pi y^2 \rho \, dx$

M.I of the disc $= \frac{1}{2} \pi y^2 \rho \, dx \, (y^2)$

$$= \frac{1}{2} \pi \rho \, y^4 \, dx$$

M.I of the Solid Cone $= \int_0^h \frac{1}{2} \pi \rho \, y^4 \, dx$

$$= \frac{1}{2} \pi \rho \int_0^h y^4 \, dx$$

ΔAEF and ΔADC are Similar triangles

$$\frac{r}{h} = \frac{y}{x}$$

$$y = \frac{xr}{h}$$

M.I of the Solid Cone $= \frac{1}{2} \pi \rho \int_0^h \frac{r^4 x^4}{h^4} \, dx$

$$= \frac{1}{2} \pi \rho \frac{r^4}{h^4} \int_0^h x^4 \, dx$$

$$= \frac{1}{2} \pi \rho \frac{r^4}{h^4} \left(\frac{x^5}{5} \right)_0^h$$

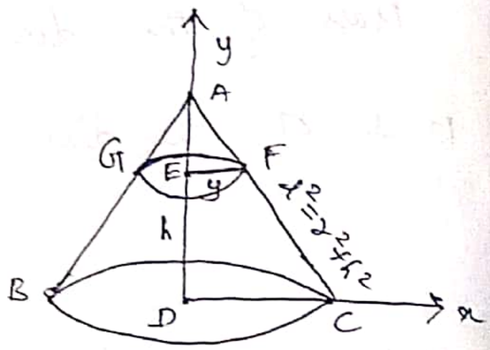
$$= \frac{1}{2}\pi\rho \frac{r^4}{h^4}\left(\frac{h^5}{5}\right)$$

$$= \frac{1}{10}\pi r^4 h \rho$$

$$= \frac{1}{10}\pi r^4 h\left(\frac{M}{\frac{1}{3}\pi r^2 h}\right)$$

M.I of the Solid Cone $= \frac{3}{10} M r^2$

8. Find the M.I of hollow Cone.



Soln:

A - vertex of Cone

h - height of Cone

x, y - Two axis along AD perpendicular to AD respectively.

Divide the Cone into a number of Circular belts of thickness ds.

Let G, F be one such belt of radius y and at a distance x from A.

Area of belt $= 2\pi y\, ds$

Mass of belt $= 2\pi y \rho\, ds$

M.I of belt $= (2\pi y \rho\, ds) y^2$

$$= 2\pi y^3 \rho\, ds$$

$$\text{M.I of hollow Cone} = \int_0^h 2\pi y^3 \rho \, ds$$

$$= 2\pi\rho \int_0^h y^3 ds$$

$$= 2\pi\rho \int_0^h \frac{r^3 x^3}{h^3} \left(\frac{l}{h}\right) dx$$

$$= \frac{2\pi\rho r^3 l}{h^4} \int_0^h x^3 dx$$

$$= \frac{2\pi\rho r^3 l}{h^4} \left(\frac{x^4}{4}\right)_0^h$$

$$= \frac{2\pi r^3 l}{h^4} \left(\frac{h^4}{4}\right)\rho$$

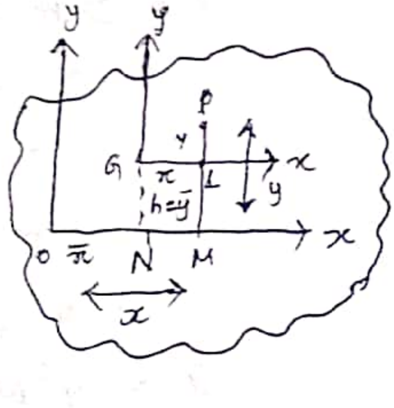
$$= \frac{2\pi r^3 l}{4} \left(\frac{M}{\pi r l}\right)$$

$$\text{M.I of hollow cone} = \frac{Mr^2}{2}.$$

Theorem of parallel axis :

If I_G is the M.I of a body about an axis through its centre of gravity and I is the M.I about parallel axis then $I = I_G + Mh^2$ where h is the distance between the two parallel axes and M is mass of the body

Proof:



* Let us assume that the body is in the form of a plane lamina
* Let ox and oy be a pair of ⊥r lines in the plane.
* Let I be the M.I about ox
* Let G be the C.G of the plane lamina.
* Let $(\bar{x}, \bar{y})$ be the Co-ordinates of G
* Let P be any point of the lamina and m be a elementary mass at P.
* Let its Co-ordinate be (x, y)
* Let Gx and Gy be a pair of lines parallel to ox and oy
* Let (x, y) be the Co-ordinates of P w.r.t the axes Gx and Gy

Now,

$$x = OM = ON + NM$$

$$= ON + GL$$

$$x = \bar{x} + x$$

$$y = MP = ML + LP$$

$$= NG + LP$$

$$y = \bar{y} + y$$

Now, M.I of the lamina about ox is

$$I = \Sigma m (PM)^2$$

$$= \Sigma my^2$$

$$= \Sigma m(\bar{y} + y)^2$$

$$= \Sigma m (\bar{y}^2 + y^2 + 2y\bar{y})$$

$$= \bar{y}^2 \Sigma m + 2\bar{y} \Sigma my + \Sigma my^2$$

$$I = \bar{y}^2 M + 0 + \Sigma my^2 \longrightarrow (1)$$

Since $\frac{\Sigma my}{m}$ is the y coordinates of the C.G of the lamina.

w.r.t axis Gx and Gy and it is zero as G is the origin

$$\Sigma my = 0.$$

Also

$$\Sigma my^2 = \Sigma m \cdot PL^2$$

$$= \text{M.I of the lamina about } Gx$$

$$\Sigma my^2 = I_G$$

Also,

$\bar{y} = h$ = distance between the axis ox and Gx

Hence from (1)

$$I = I_G + Mh^2.$$

Note:

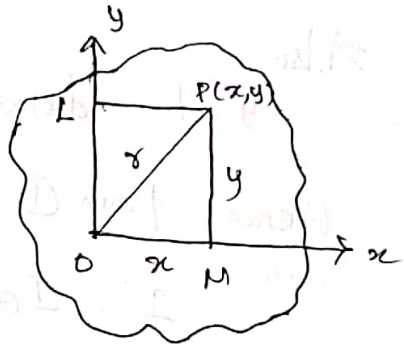
1. The above result is true in the case of any rigid body.
2. It is easily seen that the M.I about the C.G of the body is minimum compared to any axis parallel to the axis through G.

Theorem of perpendicular axis:

If I_x and I_y be the M.I of a plane lamina about two rectangular axis ox and oy in the plane of the lamina then the M.I about an axis through O $\perp$r to its plane is $I_x + I_y$

Proof:

Let ox and oy be a pair of $\perp$r lines in the plane of the lamina and oz be the axis through 'o' $\perp$r to the plane of the lamina.



Let m be an elementary mass at P. Draw PM $\perp$r to ox and PL $\perp$r to oy. then the M.I of the lamina about

$$ox = I_x = \Sigma m\, PM^2$$

The M.I of the lamina about $oy = I_y = \Sigma m\, PL^2$

Now the M.I of the lamina about oz

$$I = \Sigma m\, op^2$$

$$= \Sigma m\, (OM^2 + PM^2)$$

$$= \Sigma m\, PL^2 + \Sigma m\, PM^2$$

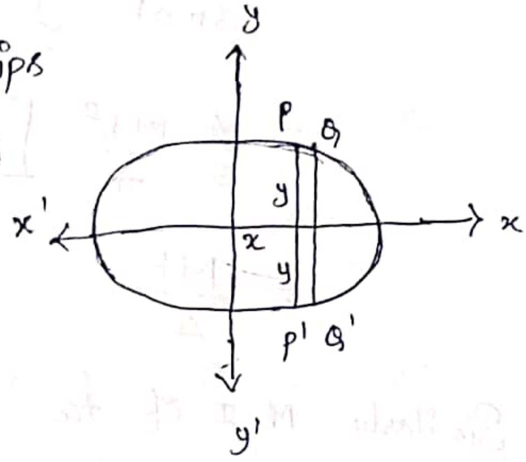
$$= I_y + I_x$$

$$I = I_x + I_y.$$

Note: The $\perp$r axis theorem holds only to a plane lamina while the parallel axis theorem holds for any rigid body.

Elliptic Lamina:

Divide the lamina into strips $PQ P'Q'$ perpendicular to major axis coordinates $P(x, y)$, $Q(x+\delta x, y+\delta y)$



$2y$ – length of the strip

dx – width of the strip

Area of the strip $= 2y\,dx$

Mass of the strip $= 2y\,dx\,\rho$

M.I of the strip about $ox = (2y\rho\,dx)\dfrac{y^2}{3}$

M.I of the elliptic lamina about $ox = \displaystyle\int_{-a}^{a} \frac{2}{3}\rho y^3 dx$

$$= 2/3\,\rho \int_{-a}^{a} y^3 dx$$

$$= \frac{4}{3}\rho \int_{-a}^{a} y^3 dx$$

put $x = a\cos\theta$, $y = b\sin\theta$

$dx = -a\sin\theta\, d\theta$

when $x = -a$, $\theta = \pi$

$x = a$, $\theta = 0$

Also $\rho = \dfrac{M}{\pi ab}$

$$M.I = 2/3\left(\frac{M}{\pi ab}\right)\int_{\pi}^{0}(b\sin\theta)^3(-a\sin\theta)\,d\theta$$

$$= \frac{2M}{3\pi ab}\int_{0}^{\pi} b^3 a\sin^4\theta\, d\theta$$

$$M.I = \frac{2M \times 2}{3\pi ab} \int_0^{\pi/2} b^3 a \sin^4\theta \, d\theta .$$

$$= \frac{4}{3} \frac{Mb^2}{\pi} \left[\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right]$$

$$= \frac{Mb^2}{4}$$

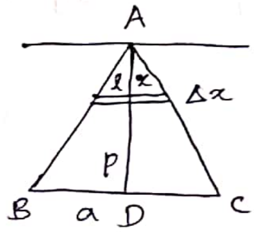
Similarly M.I of the ellipse about minner axis $(oy) = \frac{Ma^2}{4}$.

Triangular lamina:

Dividing the lamina into strips parallel to the BC at a distance x from A

Bradth : Δx

Length : $\frac{ax}{p}$ by similar triangles.



Area : $\frac{ax}{p} \Delta x$ $\rightarrow$ Mass : $\left(\frac{ax}{p} \cdot \Delta x\right)\left[\frac{M}{\frac{1}{2}ap}\right]$

Mass : $\frac{2M}{p^2} x \Delta x$

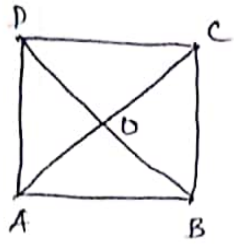
M.I about l = $\left(\frac{2M}{p^2} x \Delta x\right) x^2$

Thus the M.I of the lamina about l is

$$\frac{2M}{p^2} \int_0^p x^3 dx = \frac{2M}{p^2} \cdot \frac{p^4}{4}$$

$$= \frac{1}{2} M p^2$$

M. I of a square lamina of side $2a$ about a diagonal.



ABCD - Square lamina

AC, BD - diagonal meet at O.

O - C.G of lamina

I - M.I about the diagonal AC and BD

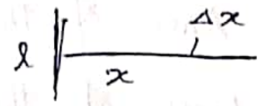
By $\perp$r axis theorem the M.I of the lamina about the line through 'O' $\perp$r to the lamina

$= I + I$

$= 2I$

M.I of a rod :

Consider an elementary length $2a$ of the rod at a distance x from l



Let its length be Δx

For the elementary length

$$\text{Mass} = \Delta x . \rho$$

$$= \Delta x . \frac{M}{2a}$$

$$\text{M.I of the about } l = \left(\frac{M}{2a}\Delta x\right)x^2$$

$$\text{M.I of the rod} = \int_0^{2a} \frac{M}{2a} x^2\, dx$$

$$= \frac{M}{2a}\left(\frac{x^3}{3}\right)_0^{2a}$$

$$= \frac{M}{2a}\left(\frac{8a^3}{3}\right) = \frac{4Ma^2}{3}$$

Note :

$$\text{M.I of the rod} = 2\int_0^a \frac{M}{2a} x^2 \, dx$$

$$= \frac{2M}{2a}\left(\frac{x^3}{3}\right)_0^a$$

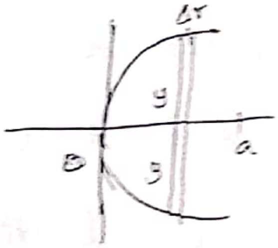
$$= \frac{M}{a}\left(\frac{a^3}{3}\right)$$

$$\text{M.I of the rod} = \frac{Ma^2}{3}$$

M.I of parabolic lamina

Divide the lamina into thin parallel strips perpendicular to the axis of the parabola.

Area of the strip $= 2y\,\Delta x$



Mass of the strip $= (2y\Delta x)\rho$

$$\left.\begin{array}{l}\text{M.I of the strip} \\ \text{about } l\end{array}\right\} = \tfrac{1}{3}(2y\Delta x)\rho\, y^2$$

$$\left.\begin{array}{l}\text{M.I of the parabolic} \\ \text{lamina}\end{array}\right\} = \frac{2\rho}{3}\int_0^a y^3\,dx$$

$$= \frac{2\rho}{3}\int_0^a 2^3(\sqrt{a})^3(\sqrt{x})^3\,dx$$

$$= \frac{16\rho a^{3/2}}{3}\int_0^a x^{3/2}\,dx$$

$$= \tfrac{16}{3}\, a^{3/2}\,\rho\left[\frac{x^{5/2}}{5/2}\right]_0^a$$

$$= \tfrac{16}{3}\times\tfrac{2}{5}\, a^{3/2}\,\rho\,[a^{5/2}]$$

$$= \frac{32}{15} a^4 \frac{M}{8/3\, a^2}$$

$$= \frac{32}{15} \times \frac{3}{8} M a^2$$

$$\left.\begin{array}{l}\text{M.I of the parabolic} \\ \text{lamina}\end{array}\right\} = 4/5\, M a^2$$

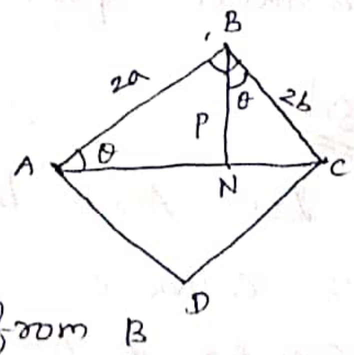
Radius of gyration :

If the M.I of a system of mass M about a straight line is written as Mk^2, then k is called the radius of gyration of the system about the straight line. For example, the radius of gyration of the case 1 to 5 are respectively $a, a, \frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}, a\sqrt{2/5}$.

Problems :

1. S.T the M.I of a rectangular lamina of mass M and sides 2a and 2b about a diagonal is $M \frac{2a^2b^2}{3(a^2+b^2)}$

Soln:



Let ABCD be the rectangle $AB = 2a$, $BC = 2b$.

Let BN be the perpendicular from B to AC and $BN = p$. If $\angle BAN = \theta$ Then

Scanned with CamScanner

$\sin\theta = \frac{p}{2a}$

$\cos\theta = \frac{p}{2b}$

$$p^2 = \frac{4a^2b^2}{a^2+b^2} \longrightarrow (1)$$

So, Now $M/2$, $M/2$ are the masses of ΔABC, ΔADC. So M.I of ΔABC about $AC = \frac{1}{6}\frac{M}{2}p^2$

M.I of ΔADC about $AC = \frac{1}{6}\frac{M}{2}p^2$

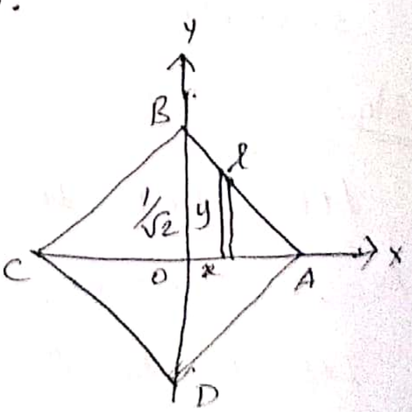
M.I of ΔABC $= \frac{1}{6}\frac{M}{2}p^2 + \frac{1}{6}\frac{M}{2}p^2$

$$= \frac{M}{6}p^2$$

$$= \frac{M}{6}\frac{4a^2b^2}{a^2+b^2}$$

$$= \frac{2Ma^2b^2}{3(a^2+b^2)} \,//.$$

2. Find the M.I of a square lamina of side l about one of its diagonals, the density at any point varying as the square of its distance from this diagonal.



Soln:

Let ABCD be the square with centre O.

Consider ΔOAB

$$OA = \frac{l}{\sqrt{2}} = \lambda \text{ (say)} \longrightarrow (1)$$

If OA, OB are x, y axes, the equation of AB is

$$x + y = \lambda$$

Divide ΔOAB into strips perpendicular to OA.

Consider a typical strip at a distance x from the diagonal BD about which we want the M.I its density is kx^2,

For this strip we have

Length : y

Area : $y \Delta x$

Mass : $y \Delta x \, (kx^2)$

M.I : $[y \Delta x \, (kx^2)] \, x^2$

$$\text{M.I of } \Delta OAB = k \int_0^{OA} y x^4 \, dx$$

$$= k \int_0^{\lambda} (\lambda - x) x^4 \, dx$$

$$= \cancel{\frac{k x^6}{30}}$$

$$= \frac{k \lambda^6}{30}$$

$$\text{M.I of the square } ABCD = 4 \cdot \frac{k\lambda^6}{30} = \frac{2}{15} k \lambda^6 \rightarrow (2)$$

$$\text{Mass of } \Delta OAB = \int_0^{OA} y \, (kx^2) \, dx = k \int_0^{\lambda} (\lambda - x) x^2 \, dx$$

$$= \frac{k\lambda^4}{12}$$

Mass M of the square ABCD $= 4 \cdot \frac{k\lambda^4}{12}$

$$= \frac{1}{3} k\lambda^4 \longrightarrow (3)$$

Dividing (2) by (3) we get,

$$M.I = M \cdot \frac{2}{15} k\lambda^6 \cdot \frac{3}{k\lambda^4}$$

$$= \frac{2M\lambda^2}{5}$$

$$= \frac{2M}{5} \cdot \frac{l^2}{2} \text{ by } (1)$$

$$M.I = \frac{Ml^2}{5}$$